X. *The Solution of* Kepler's Problem, *by* J. Machin, *Aſtr. Prof.* Greſh. *and Secr.* R. S.

MANY Attempts have been made at different times, but, if I miſtake not, never any yet with tolerable Succeſs, towards the Solution ot the Problem propoſed by *Kepler:* To divide the Area of a Semicircle into given Parts, by a Line from a given Point of the Diameter, in order to find an univerſal Rule for the Motion of a Body in an Elliptic Orbit. For among the ſeveral Methods offered, ſome are only true in Speculation, but are really of no Service. Others are not different from his own, which he judged improper : And as to the reſt, they are all ſome way or other ſo limited and confined to particular Conditions and Circumſtances, as ſtill to leave the Problem in general untouched. To be more particular ; it is evident, that all Conſtructions by Mechanical Curves are ſeeming Solutions only, but in reality unapplicable; that the Roots of infinite Series's are, upon account of their known Limitations in all reſpects, ſo far from affording an Appearance of being ſufficient Rules, that they cannot well be ſuppoſed as offered for any thing more than Exerciſes in a Method of Calculation. And then, as to the univerſal Method, which proceeds by a continued Correction of the Errors of a falſe Poſition, it is, when duly conſidered, no Method of Solution at all in itſelf; becauſe, unleſs there be ſome antecedent Rule or Hypotheſis to begin the Operation, (as ſuppoſe that of

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an uniform Motion about the upper Focus, for the Orbit of a Planet; or that of a Motion in a Parabola for the perihelian Part of the Orbit of a Comet; or some other such) it would be impoffible to proceed one Step in it. But as no general Rule has ever yet been laid down, to affift this Method, fo as to make it always operate, it is the fame in Effect as if there were no Method at all. And accordingly in Experience it is found, that there is no Rule now fubfifting but what is abfolutely ufelefs in the Elliptic Orbits of Comets; for in fuch Cafes there is no other way to proceed but that which was ufed by *Kepler:* To compute a Table for fome Part of the Orbit, and therein examine if the Time to which the Place is required, will fall out any-where in that Part. So that, upon the whole, I think, it appears evident, that this Problem (contrary to the received Opinion) has never yet been advanced one Step towards its true Solution: A Confideration which will furnifh a fufficient Plea for meddling with a Subject fo frequently handled; efpecially if what is offered fhall at the fame time appear (as I truft it will) to contribute towards fupplying the main Defect.

LEMMA I.

L E M M A I.

The Tangent of an Arch being given, to find the Tangent of its Multiple.

Let r be the Radius of the Circle, t the Tangent of a given Arch A, and n a given Number. And let T be the Tangent of the Multiple Arch $n \times A$ to be found.

Then if $\varrho\varrho$ be put for $- rr$, and $\tau\tau$ for $- tt$;

The Tangent T will be $\dfrac{\overline{r+\tau}\,\big|^{n} - \overline{r-\tau}\,\big|^{n}}{\overline{r+\tau}\,\big|^{n} + \overline{r-\tau}\,\big|^{n}} \varrho :$

Which Binomials being raifed according to Sir *Ifaac Newton*'s Rule, the fictitious Quantities τ and ϱ will difappear, and the Tangent T will become equal to

$$\dfrac{nt - \dfrac{n}{1}\cdot\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}\cdot\dfrac{t^3}{r^2} + \dfrac{n}{1}\cdot\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}\dfrac{n-3}{4}\dfrac{n-4}{5}\cdot\dfrac{t^5}{r^4} - \&c.}{1 - \dfrac{n}{1}\cdot\dfrac{n-1}{2}\cdot\dfrac{tt}{rr} + \dfrac{n}{1}\cdot\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}\cdot\dfrac{n-3}{4}\cdot\dfrac{t^4}{r^4} - \&c.}$$

This Theorem (which I formerly found for the Quadrature of the Circle, at a time when it was not known here to have been invented before) has now been common for many Years; for which Reafon I fhall premife it, at prefent, without any Proof; only for the fake of fome Ufes that have not yet been made of it.

Corollary 1. From this Theorem for the Tangent, the Sine (fuppofe) T, and Cofine Z of the Multiple Arch $n \times A$, may be readily found.

For if y be the Sine, and z the Cofine of the given Arch A, then putting vv for $- yy$, and fubftituting $\dfrac{ry}{z}$ for t, and $\dfrac{rv}{z}$ for τ, and $\dfrac{rT}{\sqrt{rr+TT}}$ for T:

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The Sine Υ will be $\dfrac{\overline{z+v}\,|^n - \overline{z-v}\,|^n}{2\,r^n}\varsigma.$

The Cosine Z will be $\dfrac{\overline{z+v}\,|^n + \overline{z-v}\,|^n}{2\,r^{n-1}}.$

Each of these may be expressed differently in a Series, either by the Sine and Cosine conjointly, or by either of them separately.

Thus Υ *the Sine* of the multiple Arch $n \times A$, may be in either of these two Forms, *viz.*

$$= \frac{z^{n-1}}{r^{n-1}} y \text{ in } n - \frac{n-1}{2} \cdot \frac{n-2}{3} A \cdot \frac{y^2}{z^2} + \frac{n-3}{4} \cdot \frac{n-4}{5} B \cdot \frac{y^4}{z^4} - \&c.$$

$$\text{or} = n y - \frac{n n - 1}{2.3\,r r} A\, y^3 - \frac{n n - 9}{4.5\,r r} B\, y^5 - \frac{n n - 25}{6.7\,r r} C\, y^7 - \&c.$$

Wherein the Letters A, B C, &c. stand, as usual, for the Coefficients of the preceding Terms.

The first of these Theorems terminates when n is any integer Number, the other (which is Sir *Isaac Newton*'s Rule, and is derived from the former by substituting $\sqrt{r r - y y}$ for z) terminates when n is any odd Number.

The *Cosine* Z may, in like manner, be in either of these two Forms, *viz.*

$$= \frac{z^n}{r^{n-1}} \text{ in } 1 - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{y^2}{z^2} + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{y^4}{z^4} - \&c.$$

$$\text{or} = r - \frac{n n}{2\,r r} A\, y^3 - \frac{n n - 4}{3.4\,r r} B\, y^4 - \frac{n n - 16}{5.6\,r r} C\, y^6 - \&c.$$

The latter of which terminates when the Number n is even, and the other as before, when it is any Integer.

Corollary 2. Hence the Sine, Cosine, and Tangent of any Submultiple Part of an Arch (suppose) $\frac{1}{n} A$, may be determined thus:

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The Tangent of $\frac{1}{n}A$ will be $\dfrac{\overline{r+\tau}|^{\frac{1}{n}} - \overline{r-\tau}|^{\frac{1}{n}}}{\overline{r+\tau}|^{\frac{1}{n}} + \overline{r-\tau}|^{\frac{1}{n}}} \varrho,$

The Sine of $\frac{1}{n} A$ will be $\dfrac{\overline{z+v}|^{\frac{1}{n}} - \overline{z-v}|^{\frac{1}{n}}}{2\,r^{\frac{1}{n}}} \varrho,$

For thefe Equations will arife from the Tranfpofition and Reduction of the former for the Tangent and Sine of the Multiple Arch, upon the Subftitution of t, y, z and A; for T, Y, Z and $n \times A$.

Corollary 3 Hence regular Polygons of any given Number of Sides may be infcribed within, or circumfcribed without, a given Arch of a Circle. For if the Number n exprefs the double of the Number of Sides to be infcribed within, or circumfcribed about, the given Arch A; then one of the Sides infcribed will be the double of the Sine, and one of the Sides circumfcribed the double of the Tangent of the Submultiple part of the Arch, *viz.* $\frac{1}{n}A$.

L E M M A II.

To find the Length of the Arch of a Circle within certain Limits, by means of the Tangent and Sine of the Arch.

Let t be the Tangent, y the Sine and z the Cofine of the Arch A, whofe Length is to be determined, and let ϱ, τ, v be expounded as before; then, if any Number n be taken, the Arch of the Circle will be

always lefs than $\dfrac{\overline{r+\tau}|^{\frac{1}{n}} - \overline{r-\tau}|^{\frac{1}{n}}}{\overline{r+\tau}|^{\frac{1}{n}} + \overline{r-\tau}|^{\frac{1}{n}}} \times n\varrho,$

and bigger than $\dfrac{\overline{z+v}|^{\frac{1}{n}} - \overline{z-v}|^{\frac{1}{n}}}{2\,r^{\frac{1}{n}}} \times n\varrho.$

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For if, by the preceding *Corollaries*, a regular rectilinear Polygon be inscribed within, and another without, the Arch A, each having half so many Sides as is expressed by the Number n; then will the former of these Quantities be the Length of the Bow of the circumscribed Polygon, (or the Sum of all its Sides) which is always bigger, and the latter will be the Length of the Bow of the inscribed Polygon, which is always less, than the Arch of the Circle; how great soever the Number n be taken.

Corollary 1. Hence the Series's for the Rectification of the Arch of a Circle may be derived.

For by converting the Binomials into the Form of a Series, that the fictitious Quantities, ρ, τ, v may be destroyed; it will appear, that no Number n can be taken so large as to make the inscribed Polygon so big, or the circumscribed so little as the Series.

$$\frac{r\,y}{z} - \frac{r\,y^3}{3\,z^3} + \frac{r\,y^5}{5\,z^5} - \frac{r\,y^7}{7\,z^7} + \&c.$$ in one Case, or its Equal

$$t - \frac{t^3}{3\,r^2} + \frac{t^5}{5\,r^4} - \frac{t^7}{7\,r^6} + \&c.$$ in the other Case.

Wherefore since the Quantity denoted by the Sum of the Terms in either of these Series's is always bigger than any inscribed Polygon, and always less than any circumscribed, it must therefore be equal to the Arch of the Circle.

Corollary 2. If, in the first of the above Series's, the Root $\sqrt{rr - yy}$, be extracted and substituted for z, there will arise the other Series of Sir *Isaac Newton*, for giving the Arch from the Sine; namely,

$$y + \frac{y^3}{6\,r^2} + \frac{3\,y^5}{40\,r} + \frac{5\,y^7}{112\,r^6} + \&c.$$ or otherwise,

$$= y + \frac{1}{1.2.3} \times \frac{y^3}{r^2} + \frac{3.3}{1.2.3.4.5} \times \frac{y^5}{r^4} + \frac{3.3.5.5}{1.2.3.4.5.6.7} \times \frac{y^7}{r^6} + \&c.$$

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S C H O L I U M.

In like manner, as the Arches of the Polygons ferve to determine the Arch of the Circle, fo by comparing the Areas of the circumfcribed and infcribed Polygons, $\frac{1}{2}nrT$ and $\frac{1}{2}nTZ$, the Area of the Sector of a Circle may be found. For if T, T and Z are the Tangent, Sine and Cofine of the Arch A; then by the fecond *Lemma* the Area of the circumfcribed Polygon

will be found to be $\frac{1}{2}nr\varsigma \times \dfrac{\overline{r+\tau}\rvert^{\frac{1}{n}}-\overline{r-\tau}\rvert^{\frac{1}{n}}}{\overline{r+\tau}\rvert^{\frac{1}{n}}+\overline{r-\tau}\rvert^{\frac{1}{n}}}=\frac{1}{2}nrT.$

and the Area of the infcribed will appear to be

$$\frac{1}{2}n\varsigma \times \dfrac{\overline{z+v}\rvert^{\frac{2}{n}}-\overline{z-v}\rvert^{\frac{2}{n}}}{4r^{\frac{2}{n}}-1}=\frac{1}{2}nTZ.$$

But upon the Expanfion of thefe Binomials it will appear, that no Number n can be taken fo large as to make the one fo big, or the other fo little, as the Area denoted by the Series.

$$\frac{1}{2}r \text{ in } t-\frac{t^3}{3rr}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\&c.$$

So that this Area being larger than any infcribed, and fmaller than any circumfcribed Polygon, muft be equal to the Area of the Sector.

It may further be obferved, that as the Arch or Area is found from the Sine, Cofine or Tangent of the Arch, by means of the limiting Polygons, fo may the Sine, Cofine or Tangent be found from the Length of the Arch by the fame Method.

Thus, if A be the Arch whofe Tangent T, Sine T, and Cofine Z, are to be determined, then will the
Tangent

$$\text{Tangent } T \text{ be} = \frac{A - \frac{1}{1.2.3} \times \frac{A^3}{r^2} + \frac{1}{1.2.3.4.5} \times \frac{A^5}{r^4} - \&c.}{1 - \frac{1}{1.2} \times \frac{A^2}{rr} + \frac{1}{1.2.3.4} \times \frac{A^4}{r4} - \&c.}$$

$$\text{Sine } T \quad = A - \frac{1}{1.2.3} \times \frac{A^3}{r^2} + \frac{1}{1.2.3.4.5} \times \frac{A^5}{r^4} - \&c.$$

$$\text{Cosine } Z \quad = r - \frac{1}{1.2} \times \frac{A^2}{r} + \frac{1}{1.2.3.4} \times \frac{A^4}{r^3} \&c.$$

For it may be made to appear, from the firſt *Lemma*, and its *Corollaries*, that if in any of theſe Theorems, as ſuppoſe in the Firſt, the Quantity A ſtand for the Bow of the circumſcribed Polygon, then will the Quantity T exhibited by the Theorem, be always bigger; but if for the Bow of the inſcribed, always leſs than the Tangent of the Arch, how great ſoever the Number n be taken; and conſequently, if A ſtand for the Length of the Arch itſelf, the Quantity T muſt be equal to the Tangent; and the like may be ſhewn for the Sine, and, *mutatis mutandis*, for the Coſine.

Theſe Principles, from whence I have here derived the Quadrature of the Circle, which is wanted in the Solution of the Problem in hand, happen to be upon another Account abſolutely requiſite for the Reduction of it to a manageable Equation. But I have inlarged, more than was neceſſary to the Problem itſelf, on the Uſes of this ſort of Quadrature by the limiting Polygons, becauſe it is one of that kind which requires no other Knowledge but what depends on the common Properties of Number and Magnitude; and ſo may ſerve as an Inſtance to ſhew that no other is requiſite for the Eſtabliſhment of Principles for Arithmetick and Geometry. A Truth, which though certain in itſelf, may perhaps ſeem doubtful from the Nature and Tendency of the preſent Inquiries in Mathematicks. For among the Moderns ſome have thought it neceſſary,

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for the Inveſtigation of the Relations of Quantities, to have recourſe to very hard Hypotheſes; ſuch as that of Number infinite and indeterminate ; and that of Magnitudes in *Statu fieri*, exiſting in a potential Manner, which are actually of no Bigneſs. And others, whoſe Names are truly to be reverenced on account of their great and ſingular Inventions, have thought it requiſite to have recourſe even to Principles foreign to Mathematicks, and have introduced the Conſideration of efficient Cauſes and Phyſical Powers for the Production of Mathematical Quantities; and have ſpoken of them, and uſed them, as if they were a Species of Quantities by themſelves.

N. B. *In the following Propoſition I have, for the ſake of Brevity, made uſe of a peculiar Notation for compoſite Numbers (or ſuch Quantities as are analogous to them) whoſe Factors are in Arithmetical Progreſſion.*

The Quantity expreſſed by this Notation has a double Index : that at the Head of the Root at the Right-hand, but ſeparated by a Hook to diſtinguiſh it from the common Index, denotes the Number of Factors, and that above, within the Hook on the Left-hand, denotes the common Difference of the Factors proceeding in a decreaſing or increaſing Arithmetical Progreſſion.

Thus the Quantity $\overline{n+a}\,{\overset{a}{(}}{}^{m}$ *denotes by its Index* m *on the Right-hand, that it is a compoſite Quantity, conſiſting of ſo many Factors as there are Units in the Number* m*; and the Index* a *above, on the Left, denotes the common Difference of the Factors, decreaſing in an Arithmetical Progreſſion, if it be Poſitive ; or increaſing, if it be Negative ; and ſo ſignifies, in the common Notation, the compoſite Number or Quantity,* $\overline{n+a}.\overline{n+a-a}.\overline{n+a-2a}.\overline{n+a-3a}.$ *and ſo on.*

For Example : $\overline{n+5}\,{\overset{2}{(}}{}^{6}$ *is* $=\overline{n+5}.\overline{n+3}.\overline{n+1}.\overline{n-1}.\overline{n-3}.\overline{n-5},$ *conſiſting of ſix Factors whoſe common Difference is* 2. *After the ſame manner* $\overline{n+4}\,{\overset{2}{(}}{}^{5}$ *is* $=\overline{n+4}.\overline{n+2}.n.\overline{n-2}.\overline{n-4},$ *conſiſting of five Factors. According to which Method it will eaſily appear, that if* a *be any Integer, then* $\overline{n+2a+1}\,{\overset{2}{(}}{}^{2a+2}$ *will be* $=\overline{nn-1}.\overline{nn-9}.\overline{nn-25},$

continued

continued to such a Number of double Factors as are expressed by a+1, *or half the Index, which in this case is an even Number. So*

$\overline{n+2a}^{(2a+1)}$ *will be equal to* n. $\overline{nn-4}$. $\overline{nn-16}$. $\overline{nn-36}$, *and so on, where there are to be so many double Factors as with one single one* (n) *will make up the Index* 2a+1, *which is an odd Number.*

If the common Difference α *be an Unit, it is omitted:*

Thus, $\overset{\frown}{n}$(6 *is* $= n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}$, *containing* 6 *Factors.*
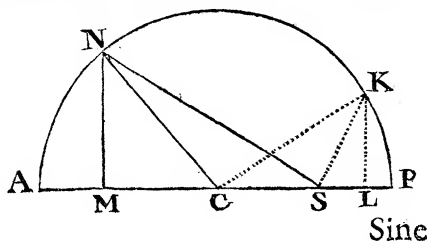
So $\overset{\frown}{6}$(6 *is* $= 6.5.4.3.2.1$, *and the like for others.*

If the common Difference α *be nothing, then the hook is omitted, and it becomes the same with the Geometrical Power:*

So $\overset{o}{\overline{n+a}}$(m *is* $= \overline{r+.}|^m$ *according to the common Notation.*

PROPOSITION I.

An Arch less than a Semicircle being given, with a Point in the Diameter passing through one of its Extremities; to find, by means of the Sine of a given Part of the Arch less than one half, the Area of the Sector subtended by the given Arch, and comprehended in the Angle made at the given Point.

Let P N A be a Semicircle described on the Centre C, and Diameter A P, and let P N be the given Arch less than a Semicircle, and S the given Point in the Diameter A P passing thro' one of the Extremities of the Arch N P in P. Then taking any Number *n* bigger than 2, let P K be an Arch in Proportion to the given Arch P N, as Unity to the Number *n*; and let it be required to find by means of the

Sine

Sine of the Arch P K, the Area of the Sector N S P subtended by the given Arch N P, and comprehended in the Angle N S P made at the given Point S.

From N and K let fall on the Diameter A P the Perpendiculars N M and K L, and join C N and C K.

Then let t stand for C P the Semidiameter of the Circle; f for C S the Distance of the given Point S from the Centre; p for S P the Distance of it from the Extremity of the Arch through which the Diameter A P passes; and y for K L the Sine of the Arch K P in the given Circle.

These Substitutions being presupposed, the Problem is to be divided into two Cases; one when S P is less, and the other when it is greater than the Semidiameter C P.

CASE I.

If S P be less than C P, then take an Area H equal to the Sum of the Rectangles expressed by the several Terms of the following Series continued *ad libitum*:

$$\frac{py}{1}+\frac{t+\overline{n+1}\times f_x)^3}{3|^3}+9t-\frac{\overline{n+1}\times f_x)^5}{5|^5}\times\frac{1}{t^4}+\frac{9\times 25t+\overline{n+5}|^6}{7|^7}\times\frac{1}{t^6}+\&c.$$

And the Area $\frac{1}{2}n\times H$ will determine the Area of the Sector N S P *ad libitum*.

For the Sector P S N, being the Excess of the Sector N C P above the Triangle N C S, will be the Difference of two Rectangles: $\frac{1}{2}$ C P×P N−$\frac{1}{2}$ C S×N M; but P N is the Multiple of the Arch P K, namely n× P K; and N M is the Sine of that multiple Arch: Wherefore if for C P be put t, for C S, f, according

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to the Suppofition; and if for PK be fubftituted:

$$\frac{y}{1}+\frac{1}{3}\times\frac{\overset{\cdot}{1}^3}{1^2}+\frac{9}{5}\times\frac{1^5}{1^4}+\frac{9\times25}{7}\times\frac{1^7}{1^6}+\&c. \text{ by } Cor. \text{ 2. } Lem. \text{ 2;}$$

and for NM:

$$\frac{ny}{1}-\frac{n.\overline{n+1}^{\frac{2}{3}}}{3}\times\frac{1^3}{1^2}+\frac{n.\overline{n+3}^{\frac{2}{3}}}{5^5}\times\frac{1^5}{1^4}-\frac{n.\overline{n+5}^{\frac{2}{3}}}{7^7}\times\frac{1^7}{1^6}+\&c. \text{ accord-}$$

ing to *Cor.* 1. *Lem.* 1. the Area of the Sector will appear in a Series, as is above determined.

But fince the Number n is greater than 2, and the given Arch PN is lefs than a Semicircle, and confequently KL or y, the Sine of the Submultiple Arch PK, is lefs than the Semidiameter CP or t; it may thence be eafily proved, that the Series will approximate to the juft Quantity of the Area, *ad libitum*.

Corollary 1. Hence, if the Number n be taken equal to $\sqrt{5+\sqrt{25+\frac{9p}{f}}}$, the Sector NSP will be

$$=\frac{1}{2}npy+\frac{n^3 t-n.\overline{nn-1}.p}{12.tt}y^3+****+\frac{n^3}{1120t^5}t^7+\&c.$$

For the Numerator of the Coefficient of the third Term in the Series, that determines the Area H, namely, $9t-\overline{n+3}^2|^4\times f$ is equal to $9t-\overline{nn-1}.\overline{nn-9}.f$, which, according to the above Determination of the Number n, will become nothing; wherefore, if for $t-p$ be put f in the fecond Term, and the Value of n be fubftituted for n in the Third and Fourth, the Series for the Area will appear upon Reduction to be as is here laid down.

Corol.

Corol. 2. Hence the Area of the Sector N S P may be always defined nearly by the Terms of a Cubic Equation.

For the Number *n*, as conftructed in the former *Corollary*, is always greater than the fquare Root of 10, and confequently $\frac{y}{t}$ is always lefs than the Sine of one third Part of the given Arch; fo that the fourth Term $\frac{n^3}{1120 t^5})^7$, with the Sum of all the following Terms of the Series, can never be more than a fmall Part of the whole Sector.

Corol. 3. If *R* ftand for 57,2957795,*&c.* Degrees, (or the Number of Degrees contained in an Angle fubtended by an Arch of the fame Length with the Radius of the Circle) and *M* be the Number of Degrees in an Angle which is to 4 right Angles, as the Area N S P to the Area of the whole Circle; then will M be $= \frac{np}{t} \times \frac{R y}{t} + \frac{n^3 t - n \cdot \overline{n n - 1} \cdot p}{6 t} \times \frac{R y^3}{t^3}$, nearly.

For $\frac{M}{R} \times \frac{tt}{2}$ will appear by the Conftruction to be equal to the Sector N S P.

C A S E II.

If S P be greater than C P, then take an Area *H* equal to the Sum of the Terms in the following Series:

$$\frac{py}{1} + \frac{t - \overline{n+1}^2 |^2 \times f}{3 |^3} \times \frac{y^3}{t^2} + \frac{9 t + \overline{n+3}^2 |^4 \times f}{5 |^5} \times \frac{y^5}{t^4} + \frac{9 \times 25 t - \overline{n+5}^2 |^6 \times f}{7 |^7} \times \frac{y^7}{t^6} + \&c$$

and the Area $\frac{1}{2}n \times H$, will be the Sector, as before.

For

For the Point S being on the contrary Side of the Centre to what it was before, it will eafily appear, that the Change of $+f$ into $-f$, muft reduce one Cafe to the other, without any other Proof.

Corollary. Hence, if the Number n be taken equal to $\sqrt{\frac{t+f}{f}}$ or in this Cafe $\sqrt{\frac{p}{f}}$ then the Series for the Sector will want the fecond Term, as in the former it wanted the Third.

<div align="center">D E F I N I T I O N.</div>

The Angle called by *Kepler* the *Anomalia Eccentri,* is a fictitious Angle in the Elliptic Orbit of a Planet, being analogous to the Area defcribed by a Line from the Centre of the Orbit, and revolving with the Planet from the Line of *Apfides;* in like manner as the *Mean Anomaly* is a fictitious Angle, analogous to the Area defcribed by a Line from the Focus.

Otherwife, if C be the Centre, S the Focus of an Elliptic Orbit defcribed on the tranfverfe Axis A P, and the Area N S P in the Circle be taken in Proportion to the whole, as the Area defcribed in the Ellipfis about the Focus, to the whole : Then is the Arch of the Circle P N, or the Angle N C P, that which *Kepler* calls the *Anomalia Eccentri.*

This Angle may be meafured either from the *Aphelion,* or from the *Perihelion;* in the following Propofition it is fuppofed to be taken from the *Perihelion.*

PRO

PROPOSITION II.

The mean Anomaly of a Comet or Planet revolving in a given Elliptic Orbit being given; to find the ANOMALIA ECCENTRI.

The Solution of this Problem requires two different Rules; the firſt and principal one ſerves to make a Beginning for a further Approximation, and the other is for the Progreſſion in approximating nearer and nearer *ad libitum.*

I. *The Rule for the firſt Aſſumption :* Let *t, f,* and *p,* ſtand as before, for the Semi-tranſverſe Axis of the Ellipſis, the Semi-diſtance of the Foci, and the *Perihelian* Diſtance; then taking the Number *n* equal to $\sqrt{5+\sqrt{25+\frac{9p}{f}}}$; let *T* ſtand for $\frac{2t}{nnt-nn-1.p}$; and *P* for $\frac{2p}{nnt-nn-1.p}$ (or $\frac{p}{t}T.$); which conſtant Numbers, being once computed for the given Orbit, will ſerve to find the Angle required nearly by the following Rule:

Let *M* be the Number of Degrees in the Angle of mean *Anomaly* to the given Time, reckoned from or to the *Perihelion*; and ſuppoſing *R,* as before, to ſtand for 57,2957, &c. Degrees; take the Number $N=\sqrt[3]{\frac{3T}{nR}}M$ and let *A* be the Angle whoſe Sine is

$$N\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{P3}{N^6}}}+N\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{P3}{N^6}}};$$

then the Multiple Angle *n* × *A* will be nearly equal to the *Anomalia Eccentri.*

The

The Truth of which will appear from the Refolution of the Cubic Equation in the laſt *Corollary* to the preceding Propoſition.

Corollary I. If the Quadruple of the Quantity $\frac{P^3}{N^6}$ be many times greater or many times leſs than Unity; or, which amounts to the ſame, if the mean *Anomaly M*, be many times leſs, or many times greater, than the Angle denoted by the given Quantity $\frac{2np}{3t} R\sqrt{P}$ (one or the other of which two Caſes moſt frequently happens in Orbits of very large Eccentricity) then the Theorem will be reduced to a ſimpler Form near enough for Uſe.

Caſe I. If M be many times leſs than $\frac{2np}{3t} R\sqrt{P}$, then the Angle A may be taken for that whoſe Sine is $\frac{t \times M}{np \times R}$.

Caſe II. If M be many times greater than $\frac{2np}{3t} R\sqrt{P}$, then let A be the Angle whoſe Sine is $N-\frac{P}{N}$; and the Multiple Angle $n \times A$, according to its Caſe, will be nearly equal to the Angle required.

Corollary II. In Orbits of very large Eccentricity, the *Perihelian* Diſtance p is many times leſs than the Semi-diſtance of the Foci f, and the Number $n = \sqrt{5+\sqrt{25+\frac{9p}{f}}}$; is always nearly equal to $\sqrt{10}$ or to the Integer 3, either of which may be uſed for it without any material Error in the Orbits of Comets.

II. *The*

II. *The Rule for a further Correction* ad libitum.

Let M be the given mean *Anomaly*, t the Semi-tranfverfe Axis, as before; and let B be equal to or nearly equal to the Multiple Angle $n \times A$ before found, then if μ be the mean *Anomaly*, and x the Planet's Diftance from the Sun, computed to the *Anomalia Eccentri B*; the Angle $\dot{B}$ taken equal to $B + \frac{t}{x} \times \overline{M - \mu}$, will approach nearer to the true Value of the Angle fought; and by Repetitions of the fame Operation, the Approximation may be carried on nearer and nearer, *ad libitum*.

This laft Rule being obvious, the Explication of it may be omitted at prefent.

S C H O L I U M.

In this Solution, where the Motion is reckoned from the *Perihelion*, the Rule is univerfal, and under no Limitation : But had the Motion been taken from the *Aphelion*, the Problem muft have been divided into two Cafes : One is, when the Eccentricity is lefs than $\frac{9}{16}$; the other is, when it is not lefs, but is either equal to, or more than in that Proportion.

If the Eccentricity be not lefs than $\frac{9}{16}$, then the fame Rule will hold, as before, only putting the *Aphelian* Diftance, fuppofe (a) inftead of the *Perihelian* Diftance (p), and fubftituting $-f$ for $+f$ in the Rule for the Number n.

If the Eccentricity be lefs than $\frac{9}{16}$, then take the Number n equal to $\sqrt{\frac{a}{f}}$, and $\frac{t}{n a} \times \frac{M}{R}$ will be nearly equal to the Sine of the Submultiple Part of the *Anomalia Eccentri* denominated by the Number n, as before.

It

It is needlefs to obferve, that the like Rules would obtain in Hyperbolic Orbits, *mutatis mutandis*. But that which perhaps may not appear unworthy of being remarked, concerning this fort of Solution from the Cubic Root, is, that although the Rule be altogether impoffible, upon a total Change of the Figure of the Orbit either into a Circle, or into a Parabola; yet it will operate fo much better, and ftand in need of lefs Correction, according as the Figure advances nearer in its Change towards either of thofe two Forms.

That the Ufe of the Method may better appear, it may not be amifs to add a few Examples.

I have given two for the Orbits of Planets, one the moft, and the other the leaft Eccentrick; but which are more to fhew the Extent of the Rule, than to recommend the Ufe of it in fuch Cafes; for there are many other much better and more expeditious Methods in Orbits of fmall Eccentricity. The other two Examples are adapted to the Orbits of two Comets, whofe Periods have been already difcovered by Dr. *Halley*; the one is to fhew the Ufe of one of the Rules in the firft *Corollary*, and the other is to explain the Ufe of the other Rule.

Example I.
For the Orbit of Mercury.

If an Unit being put for the Semi-tranfverfe Axis (t), the Eccentricity $0,20589$ will become (f), and the *Perihelian* Diftance (p) will be $0,79411$; wherefore by means of the Number R given as before, the conftant Numbers for this Orbit will appear to be,

$n = 3,56755$, $T = 0,5857271$, $P = \frac{p}{t} T = 0,4651319$, and hence

$\frac{3\,T}{n \times R} = 0,0085965$.

Example.

Example. Suppofe M the mean *Anomaly* from the *Perihelion* to be 120°. 00'. 00'', to which it is required to find the *Anomalia Eccentri.*

Here, fince the mean *Anomaly* M is not many times more than the limiting Angle $\frac{2np}{3t}R\sqrt{P}$, (which in this Orbit is about 74 Degrees) recourfe muft be had to the general Rule in the Propofition.

The Number N then, which is $\sqrt[3]{\frac{3T}{nR}}M$ will be $= 1,0104195$; which found gives

$$N\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}} = 1,0389090 \text{; and alfo}$$

$$N\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}} = -0,4477126.$$ Wherefore the Sum of both (under their proper Signs) *viz.* 0,5911964 will be the Sine whofe Arch 36°,24195 is the Angle A; the Multiple whereof $n \times A$ $= 129°,295503$, will be the Angle to be firft affumed for the *Anomalia Eccentri.*

For a further Correction; this Angle, now called B, whofe Sine is fuppofe y, and its Cofine z, gives, by a known Rule, $t + \frac{f}{t}z = 1,1304$ for x the Planet's Diftance from the Sun; and by another known Rule $B - \frac{fR}{tt}y = 120°,16568$ for μ the mean *Anomaly* to the *Anomalia Eccentri B.* Wherefore the correct Angle $B = B + \frac{t}{x} \times \overline{M-\mu}$ will be 129°,14846 $= 129°. 08'. 54''. 5$, erring, as will appear from a further Correction, about $\frac{1}{10}$ of a Second.

This

This Angle, being thus determined, will give by the common Methods $137^{\circ}.48'.33''\frac{1}{2}$, for the true *Anomaly* or Angle at the Sun: The Sine of the true *Anomaly* being in Proportion to the Sine of the *Anomalia Eccentri*, as the Semi-conjugate Axis to the Planet's Diftance from the Sun. So that the Equation of the Centre in this Example is 17° $48'. 33''\frac{1}{2}$.

E X A M P L E   II.

For the Orbit of Venus.

Suppofing, as before, the mean Diftance t to be Unity, and the Eccentricity f to be $0,0069855$; the conftant Numbers for this Orbit will be, $p = 0,9930115$; $n = 6,4116$; $T = 1,562134$; $P = 0,1551217$; $\frac{3T}{nR} = 0,0127571$; and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$, will appear to be about 303 Degrees.

Example. Let M be $120^{\circ}.00'.00''$, as in the former Example. Then, fince the mean *Anomaly* is, in this Cafe, not many times lefs than the limiting Angle, the general Rule muft be ufed as before; according to which the Number N will appear to be $1,15258.5$; the Sine of A will be $0,3217917$; the Angle A, $18^{\circ},77132$; and the Multiple $n \times A$, or Angle B; for the firft Affumption of the *Anomalia Eccentri* will be $120^{\circ},35416$.

This Angle B will give, by the Method before explained, the Angle $\overset{/}{B} = 120^{\circ},34555$; or $120^{\circ}. 21'. 44'' \textit{ferè}$, for the *Anomalia Eccentri* correct; the Error of which will appear, upon Examination, to be but a fmall Part of a Second.

In

In this Example the true *Anomaly* is 120°. 41'. 25″,1 ; and confequently the Equation of the Centre no more than 41'. 25″,1.

E x a m p l e   III.

For the Orbit of the Comet of 1682.

To know the mean *Anomaly* of this Comet to any given Time, it is to be premifed, that it was at the *Perihelion* in the Year 1682, on the 4th Day of *September*, at 21 Ho. 22 Min. equated Time to the Meridian of *Greenwich*, and makes its Revolution about the Sun, as Dr. *Halley* has difcover'd, in $75\frac{1}{2}$ Years.

The *Perihelian* Diftance p is, according to his De-termination, 0,0326085 Parts of the mean Diftance t. So that the conftant Numbers for the Orbit will be, $n = 3,1676061$; $T = 0,2054272$; $P = 0,00669867$; and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$ will be about 19 Minutes or $\frac{1}{3}$ of a Degree.

In the Orbits of Comets, the Rule for the firft Af-fumption of the *Anomalia Eccentri* is generally fuf-ficient without Correction.

Thus, fuppofe the mean *Anomaly M* to be 0,072706, (as it was at the time of an Obfervation made at *Green-wich* on the 30th of *Auguft* 1682, at 7^h. 42'. Æq. T.) then the general Rule (which muft be here ufed, fince the Angle of mean *Anomaly* is not above 4 or 5 times lefs than the limiting Angle) will give $n \times A$ or $B = 2°$ 12'. 48″,7, erring about $\frac{7}{10}$ of a Second from the true *Anomalia Eccentri*.

But

But in thefe Orbits the Rules in the firft *Corollary* to the fecond Propofition moft frequently take Place, efpecially the laft; and the Calculation may alfo be further abreviated, by putting the fquare Root of 10, or the Integer 3, for the Number *n*.

Example. Suppofe the mean *Anomaly* to be $0^o,006522$, or $23'',4792$: Here, fince M is 50 times lefs than the limiting Angle, the Rule in the firft Cafe of the firft *Corollary* may be ufed; that is, to take the Sine of the Angle $A = \frac{t \times M}{np \times R}$.

Wherefore, if the Number 3 be put for *n*, the Sine of A, which is $\frac{tM}{3pR}$, will be $= 0,00116367$; and confequently the Angle A will be $4'.00'',011$; and the multiple Angle $n \times A$ to be affumed for the *Anomalia Eccentri* will be $12'.00'',033$, the Error of which will be found to be about $\frac{1}{30}$ of a Second.

E X A M P L E   IV.

For the Orbit of the great Comet *of the Year* 1680.

This Comet, according to Dr. *Halley*, performs its Period in 575 Years; and was in its *Perihelion* on the 7th of *December* 1680, at $23^h.09'$ Æq. T. at *London*; the *perihelian* Diftance p is $0,000089301$, in Parts of the mean Diftance t: Wherefore fuppofing the Number n to be $\sqrt{10}$, the conftant Numbers for the Orbit will be $T = 0,2000161$; $P = 0,000017862$, and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$ will be about $\frac{1}{6}$ of a Second.

Example. Suppofe the mean *Anomaly* to be $3',31'',4478$ or $0^o,05873541$, (as it was at the Time of the firft Obfervation made on it in *Saxony*, on
November

November the 3d, at 16ʰ. 47′ Æq. T. at *London*.) here, fince the mean *Anomaly* is many times greater than $\frac{1}{6}$ of a Second, the Rule in the fecond Cafe of the firft *Corollary* may be ufed; that is, by taking the Sine of $A = N - \frac{P}{N}$.

But the Number N or $\sqrt[3]{\frac{3\,T}{n\,R}} M$ is $= 0,05794134$; and $\frac{P}{N}$ will be $= 0,0030827$; wherefore

$(N - \frac{P}{N} =)$ $0,05763307$, will be the Sine whofe Arch $3^{\circ},30397$ is the Angle A; and the multiple Angle $n \times A = 10^{\circ}.26'.53'',05$, will be the Angle to be firft affumed for the *Anomalia Eccentri*; the Error of which will be found to be lefs than a Second.

The true *Anomaly*, computed from this Angle according to the Rule in the Example for *Mercury*, will appear to be $171^{\circ}.38'.24''$. from the *Perihelion*.

By thefe Examples it appears, that the Solution is univerfal in all refpects; for the two firft, compared with the two laft, ferve to fhew that it is not confined to any particular Parts of the Orbit, but extends to all Degrees of mean *Anomaly*: And by comparing the fecond with the laft, it fufficiently appears to be univerfal with refpect to the feveral Degrees of Eccentricity; fince in one the Equation of the Centre for the Reduction of the Mean to the true Motion is not fo much as the $\frac{1}{170}$th Part of the whole; whereas in the other it amounts to almoft 3000 times as much as the mean Motion itfelf.

POSTSCRIPT.

UPON reviewing the Reflections on the Quadrature of the Circle in Page 212, I believe it may be necessary for me, to prevent any Mistake that may arise from the different Opinions that obtain about the Nature of Mathematical Quantity, to explain myself a little upon that head; as also to add a few Words to shew how the Method of Quadrature by limiting Polygons, takes place in other Figures as well as the Circle.

I take then a Mathematical Quantity, and that for which any Symbol is put, to be nothing else but Number with regard to some Measure which is considered as one. For we cannot know precisely and determinately, that is, mathematically, how much any thing is, but by means of Number. The Notion of continued Quantity, without regard to any Measure, is indistinct and confused; and although some Species of such Quantity, considered physically, may be described by Motion, as Lines by Points, and Surfaces by Lines, and so on; yet the Magnitudes or Mathematical Quantities are not made by that Motion, but by numbering according to a Measure.

Accordingly, all the several Notations that are found necessary to express the Formations of Quantities, do refer to some Office or Property of Number or Measure; but none can be interpreted to signify continued Quantity as such.

Thus some Notations are found requisite to express Number in its ordinal Capacity, or the *Numerus Numerans*, as when one follows or precedes another, in the first, second or third Place from that upon which it depends; as the Quantities $\dot{x}, \ddot{x}, x, \acute{x}, \acute{\acute{x}}$, referring to the principal one x.

So, in many Cases, a Notation is found necessary to be given to a Measure as a Measure; as for Instance, Sir *Isaac Newton*'s Symbol for a Fluxion $\dot{x}$; for this stands for a Measure of some kind, and accordingly he usually puts an Unit for it, if it be the principal one upon which the rest depend.

So some Notations are expresly to shew a Number in the form of its Composition, as the Index to the Geometrical Power x^n denoting the Number of equal Factors which go to the Composition of it, or what is analogous to such.

But that there is no Symbol or Notation but what refers to discrete Quantity, is manifest from the Operations, which are all Arithmetical.

And hence it is, there are so many Species of Mathematical Quantity as there are Forms of composite Numbers, or Ways in the Composition of them; among which there are two more eminent for their

Simplicity

Simplicity and Univerſality than the reſt: One is the *Geometrical Power* formed from a conſtant Root; and the other, though well known, yet wanting a Name as well as a Notation, may be called the *Arithmetical Power*; or the Power of a Root uniformly increaſing or diminiſhing, and is that whoſe Notation is deſigned in Page 213: The one is only for the Form of the Quantity itſelf, the other is for the Conſtitution of it from its Elements.

Now from the Properties of either of theſe it would be eaſy to ſhew how the Quadratures of ſimple Figures are deducible from the Areas of their limiting Polygons. I ſhall juſt point out the Method from the Arithmetical Power, as being the ſhorteſt and readieſt at hand.

Let $z, \dot{z}, \ddot{z}$ &c. or $z, \dot{z}, \ddot{z}$, &c. be Quantities in Arithmetical Progreſſion, diminiſhing or increaſing by the common Difference $\dot{z}$, and let, as before explained, $\overset{\dot{z}}{z}^{(m}$ ſignify the Arithmetical Power of z, denominated by the potential Index m, namely, $z \times \dot{z} \times \ddot{z}$, &c. whoſe firſt Root is z and laſt $z - \overline{m-1} \times \dot{z}$; which being ſuppoſed, the Element of the Arithmetical Power will be $m z \times \dot{z}$. $\overset{\dot{z}}{}^{(m-1}$ that is, the Product made from the Multiplication of the two Indices, and the next inferior Power of the next Root in Order. For the firſt Arithmetical Power $\overset{\dot{z}}{z}^{(m}$ is $= z . \overset{\ddot{z}}{z}^{(m-1}$, and the next $\overset{\dot{z}}{z}^{(m}$ is $= \overset{\dot{z}}{z}^{(m-1} \times z - m \dot{z}$, wherefore the Difference will be as is explained.

And conſequently, ſince the Sum of theſe Elements or Differences, taken in order from the firſt to the laſt, do make up the Quantity according to its *termini*; hence, it x be the Abſciſs of a curvilinear Figure whoſe Ordinate y is equal to $m z^{m-1}$; a Demonſtration might eaſily be made that the [Form of the Quantity for] the Area will be z^m; that is, the ſame Multiple of the next ſuperior Power of z divided by the Index of that Power.

For ſince the Arithmetical Powers do both unite and become the ſame with the Geometrical Power, when the differential Index $\dot{z}$ is ſuppoſed to be nothing; the Magnitude of the Geometrical Figure will be implied from the Magnitudes of the two Polygons made up of Rectangles, one from the increaſing Arithmetical Power, the other from the diminiſhing, although it be true, that the Elements of the Polygons cannot be ſummed up, when $\dot{z}$, the Meaſure of the Abſciſs z, is ſuppoſed to be nothing.

In like manner, in any other Cafe where z and $\dot{z}$ are two Abfciffes

whofe Difference as a Meafure is $\dot{z}$; and y, $\dot{y}$ the two Ordinates; the Magnitude of the Figure will be implied by the Magnitudes of the two Polygons which are made from the Sum of the infcribing and

circumfcribing Elements $\dot{z}\,y$ and $\dot{z}\,\dot{y}$, although the Figure itfelf is not to be refolved into any fuch primogenial rectangular Elements.

And thus, I think, the Symbol $\dot{z}$, confidered as a component Part of the Rectangle $\dot{z}\,y$, may bear a plain Interpretation; *viz.* that it is the Meafure according to which the Quantity z is meafured; nor can I fee that any other Interpretation need to be put upon a Symbol, which, like a Meafure, is ufed only to make other things known, but is of itfelf for nothing but a Mark.

And what is faid of the Elements of the firft Refolution, is eafily applied to thofe of a fecond or third, and fo on; the laft may always be confidered as the Meafure of the former and indivifible, although, in refpect of the following, it be taken as the Part according to which the Meafure was made, and therefore divifible.

The candid Reader is defired to ftrike out a Remark of mine, about the Comet of 1556, fubjoined to an Obfervation of the late Comet made at Lisbon, printed in Page 123 of the laft Tranfactions, Nᵒ 446. The Note was there inferted by fome Accident, without my Intention; for I had foon afterwards informed the Society, that the Remark was ill-founded. According to Mr. Bradley's Obfervations at Oxford, which were not then communicated, the Place at the Time mentioned ought to have been in Long. ♓. 13°. 21′. ½. Lat. 0°. 29′. South. So that in all probability there happened fome Miftake in making this Obfervation.

ERRATA. P. 213. L. 31. for $n.n - 2$ read $n.n - 2$.
P. 222. L. 26. for *being* read *be.*

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